2nd finite diff. expression for $u''(t)$.

$$(1) \qquad u''(t) = \frac{u(t-\Delta t) - 2u(t) + u(t+\Delta t)}{\Delta t^2} + O(\Delta t$$

In mass-spring systems:

$$(2) \qquad u''(t) = -\frac{k}{m}u(t) \qquad \text{(exact)}$$

(1) with (2) gives: $(3)\ u(t+\Delta t) = -\frac{k\Delta t^2}{m}u(t) + 2u(t) - u(t-\Delta t$

Problem: in Yade (and most DEM codes) $u(t-\Delta t)$ is already erased when you compute $u(t+\Delta t)$.

BUT: you can re-write (3):

$$u(t+\Delta t) = -\frac{k\Delta t^2}{m}u(t) + u(t) + \Delta t\left(\frac{u(t) - u(t-\Delta t)}{\Delta t}\right)$$

$$= u(t) + \Delta t\,\underbrace{\left[\underbrace{-\frac{k}{m}\Delta t(u(t))}_{} + \underbrace{\frac{u(t) - u(t-\Delta t)}{\Delta t}}_{}\right]}_{}$$

this is exactly $u''(t) \times \Delta t$

represents mean velocity on previous interval (exact), which is also a good (2nd order) approx. of $u'(t-\frac{\Delta t}{2})$

Let us call this "velocity" and store it...

all this is

$u'(t-\frac{\Delta t}{2}) + u''(t)\Delta t$

wich is an approximation (2nd or

of $u'(t+\frac{\Delta t}{2})$

Time-step:

A recursive scheme like $u(n+1) = \alpha u(n) + u(n-1)$ is converging if and only if $|\alpha| < 1$.

Here $\alpha = 2 - \frac{k\Delta t^2}{m}$

Stable $\Leftrightarrow$ $\Delta t < \sqrt{\frac{2m}{k}}$

BUT in multiple mass systems, the highest frequency is when masses have opposite motion, which gives a virtual stiffness $\tilde{k} = 2k$

So that $\boxed{\Delta t < \sqrt{\frac{m}{\tilde{k}}}}$